Calculations of trajectories with constant drag coefficient

Very fast flying projectiles with a large ratio of mass to cross-sectional area - such as **long rod penetrators** - have at the operational range a drag coefficient which can be regarded as constant. As the vertex of the trajectory is very small, also the air density can be assumed to be constant and the vertical velocity is negligible. In this case, the ballistic differential equation for the horizontal velocity leads to the following relation:

$$v = v_0 \cdot e^{-\frac{c_w \cdot \rho \cdot \pi \cdot D^2}{8 \cdot m} \cdot s}$$

s is the length of a flat trajectory and can be replaced by the horizontal distance x. By substituting all constant values, one obtains a simple equation for the velocity in function of the horizontal distance:

$$a = \frac{c_w \cdot \rho \cdot \pi \cdot D^2}{8 \cdot m} \quad \rightarrow \quad \boxed{v = v_0 \cdot e^{-a \cdot x}}$$

From given or measured velocities at the beginning and the end of the trajectory, the parameter **a** can be calculated.

$$a = \frac{1}{x_e} \cdot \ln \frac{v_0}{v_e}$$

If these velocities belong to a different air density than postulated in the Firing Table (FT), **a** has to be adjusted proportionally to these air densities.

$$a_{FT} = \frac{\rho_{FT}}{\rho} \cdot a = \frac{\rho_{FT}}{\rho} \cdot \frac{1}{x_e} \cdot \ln \frac{v_0}{v_e}$$

Equations for values in the **Firing Table** as a function of the firing distance:

1. Quadrant Elevation $\quad \varphi = \frac{g}{2 \cdot x} \left(\frac{3}{2 \cdot a_{FT} \cdot v_0} \cdot \left(e^{\frac{2}{3} \cdot a_{FT} \cdot x} - 1 \right) \right)^2$

   The dimension of φ is in radians. Multiplied with $3200/\pi$ you get mils.

2. Time of Flight $\quad t = \frac{1}{a_{FT} \cdot v_0} \left(e^{a_{FT} \cdot x} - 1 \right)$

3. Actual Velocity $\quad v = v_0 \cdot e^{-a_{FT} \cdot x}$

4. Vertex Height $\quad y = \frac{1}{8} \cdot g \cdot t^2$

5. Deviation by Crosswind $\quad z = w_{cross} \cdot \left(t - \frac{x}{v_0} \right)$

Calculation example for a row of the Firing Table

Given values:

Muzzle velocity	1630	mps
Terminal velocity	1472	mps
Range related to terminal velocity	3000	m
Air density related to the given values	1.150	kg/m^3

Parameters as basis for the firing table:

Muzzle velocity	1650	mps
Air density for the F.T.	1.225	kg/m^3
Range row	2400	m
Crosswind	10	mps

Calculation of a_{FT}

$$a_{FT} = \frac{\rho_{FT}}{\rho \cdot x_e} \cdot \ln\frac{v_0}{v_e} = \frac{1.225}{1.15 \cdot 3000} \cdot \ln\left(\frac{1630}{1472}\right) = 3.620E-05$$

Calculation of selected row values

Range x = 2400m

1. Quadrant Elevation

in radians
$$\varphi = \frac{9.81}{2 \cdot 2400}\left(\frac{3}{2 \cdot 3.62E-05 \cdot 1650} \cdot \left(e^{\frac{2}{3} \cdot 3.62E-05 \cdot 2400} - 1\right)\right)^2 = 4.583E-3$$

in mils
$$\varphi = \frac{3200}{3.14159} \cdot 4.58311E-03 = 4.67$$

2. Time of Flight
$$v = 1650 \cdot e^{-3.62E-05 \cdot 2400} = 1512.7$$

3. Actual Velocity
$$t = \frac{1}{3.62E-05 \cdot 1650}\left(e^{3.62E-05 \cdot 2400} - 1\right) = 1.520$$

4. Vertex Height
$$y = \frac{1}{8} \cdot 9.81 \cdot 1.52^2 = 2.83$$

5. Deviation by crosswind (10 mps)

$$z = 10 \cdot \left(1.520 - \frac{2400}{1650}\right) = 0.65$$